An interesting application of systems theory: the sliding of a ball on an inclined plane

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An inclined plane and a ball free to slide on it offer an interesting example of application of systems theory. Let us imagine to lead the motion s of the ball by controlling the inclination θ of the plane.

Given Newton's second law

$$m\ddot{s} = a_A - b\dot{s}$$

where $a_A = \sin\theta$ and b is the coefficient of air friction, we assume the following input and output:

$$\begin{cases} x_1 = s \\ x_2 = \dot{x_1} \\ u = a_A \\ y = x_1 \end{cases}$$

.

We can now consider the continuous-time system

$$\begin{cases} \dot{x_1}(t) = x_2(t) \\ \dot{x_2}(t) = -\frac{b}{m}x_2(t) + \frac{1}{m}u(t) \\ y(t) = x_1(t) \end{cases} \quad \begin{cases} \dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & -\frac{b}{m} \end{bmatrix} x + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} u \\ y = \begin{bmatrix} 1 & 0 \end{bmatrix} x \end{cases}$$

with the following transfer function:

$$W(s) = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} s & -1 \\ 0 & s + \frac{b}{m} \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} = \frac{1}{ms\left(s + \frac{b}{m}\right)}$$

We have an accelerometer which sends informations about acceleration every T seconds. Every time the accelerometer sends an interrupt to the computer, we reload the position of the ball and repaint the screen. It occurs to discretize the above continuous-time system:

$$W(z) = \frac{z-1}{z}\mathcal{Z}\left[\mathcal{L}^{-1}\left(\frac{W(s)}{s}\right)\Bigg|_{t=kT}\right] = \frac{z-1}{z}\mathcal{Z}\left[\mathcal{L}^{-1}\left(\frac{1}{ms^2\left(s+\frac{b}{m}\right)}\right)\Bigg|_{t=kT}\right] =$$

$$= \frac{z-1}{z}\,\mathcal{Z}\left[-\frac{m}{b^2} + \frac{1}{b}\,t + \frac{m}{b^2}\,e^{-\frac{b}{m}t}\Bigg|_{t=kT}\right] == -\frac{m}{b^2} + \frac{T}{b}\,\frac{1}{z-1} + \frac{m}{b^2}\,\frac{z-1}{z-e^{-\frac{b}{m}T}}$$

The realization of the $W(z)$ in observable canonical form is:

$$A = \begin{bmatrix} 0 & 1 \\ -\lambda & \lambda+1 \end{bmatrix} B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} C = \begin{bmatrix} m - m\lambda - Tb\lambda & -m + m\lambda + Tb \end{bmatrix}$$

Now we have the following discrete-time system which can be easily implemented in a program:

$$\begin{cases} x_1(t+1) = x_2(t) \\ x_2(t+1) = -\lambda x_1(t) + (\lambda+1)x_2(t) + u(t) \\ y(t) = \frac{m-m\lambda+Tb\lambda}{b^2}\, x_1(t) + \frac{-m+m\lambda+Tb}{b^2}\, x_2(t) \end{cases}$$

where $\lambda = e^{-\frac{b}{m}T}$